Exploring Black Holes, Chapter 3 Exercises

Table of Contents

[Problem 1. Plunging from Rest at Infinity 2](#_Toc69379340)

[Part A. 2](#_Toc69379341)

[Part B 3](#_Toc69379342)

[Part C 3](#_Toc69379343)

[Part D 4](#_Toc69379344)

[Problem 2 5](#_Toc69379345)

[Part A 5](#_Toc69379346)

[Part B 7](#_Toc69379347)

[Problem 5 8](#_Toc69379348)

[Part A 8](#_Toc69379349)

[Part B 9](#_Toc69379350)

[Part C & D 9](#_Toc69379351)

[Problem 9 10](#_Toc69379352)

[Part A 10](#_Toc69379353)

[Part B 10](#_Toc69379354)

[Part C 12](#_Toc69379355)

[Problem 10 13](#_Toc69379356)

[Appendix 14](#_Toc69379357)

[Deriving the velocity equation from the Bookkeeper’s View 14](#_Toc69379358)

[Deriving the velocity equation with a non-zero initial velocity 15](#_Toc69379359)

[Deriving the velocity equation from the Shell’s view 17](#_Toc69379360)

# Problem 1. Plunging from Rest at Infinity

Black Hole Alpha has a mass of and a horizon at . A stone starting from rest far away falls radially into Black Hole Alpha.

## Part A.

At what velocity does a shell observer at measure the stone to be going as the stone passes them? What is the bookkeeper velocity of the stone as it passes ?

To measure the speed of the stone as it passes the shell observer at , we use the work from Deriving the velocity equation from the Shell’s view to obtain the following expression:

Substituting in the values of and we evaluate to find the following:

Because the stone is just in-falling from starting a rest, we can use the work from the Deriving the velocity equation from the Bookkeeper’s View section to obtain an expression for velocity as a function of r-coordinate:

In this case, we take the negative sign, as it matches the physical situation: corresponds to a decreasing radial component and corresponds to continued forward evolution of time. We can then substitute in the r-coordinate of and the mass of the blackhole into this expression and evaluate:

## Part B

At what velocity does a shell observer, at , measure the stone to be going as it passes him? What is the bookkeeper velocity of the stone as it passes ?

For the both the shell observer and bookkeeper, we use the same equations that we have already derived:

|  |  |
| --- | --- |
| Shell Observer | Bookkeeper |
|  |  |

## Part C

Qualitatively, what do the formulas in the text lead you to expect about:

* the relative shell speeds (greater or smaller) at the two radii?

Comparing the shell observer’s formula to the bookkeeper’s, we can see that the bookkeeper’s (measurement of the) velocity (for the stone) is just a scaled version of the shell observer:

As approaches , the proportionality factor (for the bookkeeper) is going to approach , while the shell approaches . It is important to remember that these real numbers will be constrained inside the interval of .

We must be careful, however, as the shell’s formula would lead you to assume that the stone is traveling proportional to the speed of light by a factor of at , but the stone has mass which makes this conclusion unphysical.

This is due to the coordinate system that we are using which has a singularity at .

* the relative values of the shell and bookkeeper speeds (greater or smaller) at each radius?

As stated above, we expect the bookkeeper’s measured speeds to be smaller than the shell observer’s measured speed by a factor of . At large values of , the shell observer will measure a smaller speed than the bookkeeper for the values of that cause .

## Part D

In the limit as , what is the shell speed of the stone? What is the bookkeeper speed of the stone?

As we mentioned in Part C, the speed of the stone as is measured to be the extreme possible values: and .

|  |  |
| --- | --- |
| Shell Observer | Bookkeeper |
|  |  |

# Problem 2

## Part A

A stone is released from rest far from a black hole of mass . The stone drops radially inward. Bookkeeper records show that the stone’s inward speed initially increases but declines toward zero as the stone approaches the horizon. Bookkeeper speed must therefore reach a maximum at some intermediate radius . Find this radius for maximum bookkeeper speed. Check your answer using Figure 5.

We can either approach this problem graphically or algebraically. Using algebra and calculus, we can describe the point of inflection for this equation by taking the derivative and finding the critical points.

|  |  |
| --- | --- |
| Taking the second time derivative of the bookkeeper’s velocity formula: |  |
| First, let us relabel these quantities to make the product rule simpler: | Let: |
| Then, expanding the product rule, we obtain: |  |
| Undoing the substitution and simplifying: |  |
| Next, we substitute in what is and simplify further: |  |
| Now we can find the critical points by setting the left-hand side equal to 0.  We note that the only time this expression goes to are the two following cases: |  |

We know that physically, , represents the event horizon of our black hole, so the point is where the bookkeeper will observe the maximum speed for the object free-falling (radially) into the black hole. This result is consistent with Figure 5 on page 3-16.

## Part B

Find the radius of maximum bookkeeper speed for the more general case of a stone hurled into the black hole. Verify that your result reduces to the dropped-from-rest expression when the initial speed is zero.

First, we begin like the work done in the Deriving the velocity equation from the Bookkeeper’s View section but this time, instead of the stone being at rest, we give it some initial velocity. Let us denote this non-zero radial inward velocity . However, instead of having the ratio of energy to its mass being unitary, we instead have it equal to some constant value.

Returning to our Chapter 1 concept of energy per mass, we can use this to describe the speed of our stone. Special Relativity tells us that this value is :

We slightly modify this expression to account for the presence of a massive object[[1]](#footnote-1) and obtain:

We can now proceed as we did before in but with an additional term upfront. This work is carried out in the Deriving the velocity equation with a non-zero initial velocity section and we obtain the following expression (which is consistent with Sample Problem 3 on Page 3-25):

For completeness, we examine the limiting case as :

# Problem 5

A stone is hurled radially inward toward a black hole from a great distance, with initial stretch factor . Sample Problem 3 describes the shell and bookkeeper speeds of this stone after it falls inward to reduced circumference . Now hurl the stone inward with greater and greater initial energy so that the remote stretch factor approaches infinity. In other words, in the limiting case let the stone take on the properties of a flash of light.

## Part A

Show that the resulting velocities of light reckoned by the bookkeeper and measured directly by the shell observer have the values, respectively,

The second of these results seems reasonable enough. But the first expression looks strange! Investigate further.

Using our work from Problem 2: Maximum Bookkeeper Speed, Part B, we obtained the following equation to describe the speed of an in-falling stone with non-zero initial velocity:

Evaluating the limit as :

Noting that the derivation for the velocity of the in-falling stone with non-zero initial velocity from the shell observer follows the same form, we just cite the equation as written in the book:

Which, by inspection, we can see will match the given expression for the speed of light as observed by a shell observer.

## Part B

What is the “bookkeeper radial velocity of light” very far from the black hole

Taking our velocity equation, we follow the same process but instead take the limit as approaches infinity and hope to recover results from Special Relativity:

Evaluating this expression for light, we take the limit as :

We then consider this light as being “very far” from the black hole and we recover that the speed of light is unity (as expected):

## Part C & D

Does the “bookkeeper radial velocity of light” increase or decrease in magnitude as one nears the black hole? What is this “velocity” in the limiting case of approaching the horizon ()?

Resuming with the velocity equation when evaluated for a light-like object, we had obtained the following expression (in the limit as ):

Which tells us, that as , the speed of light appears not only to depend on radial distance but also drops to as it reaches the horizon! This result seems to violate one of Special Relativity’s postulates: The speed of light is constant for all observers!

Briefly reading page 5-3 and 5-4 as recommended, this is not so much a fluke or abuse of the model spitting out false information – This has been verified by experiments and apparently helps resolve the shell observer measuring an in-falling object reach the speed of light!

# Problem 9

A robot worker on the shell at drops a tool from rest. What initial acceleration will the robot measure for the tool? Answer this question using the following outline or some other method.

## Part A

Express Newtonian acceleration in geometric units. According to Newton, what is the radial acceleration at a distance from a spherically symmetric center of attraction, in conventional units? Express the result as , where *conv* is shorthand for *in conventional units*. Then set and show that the Newtonian prediction, expressed in geometric units is

While ignoring that acceleration is in the negative radial direction. In these units, what is the value of , the acceleration of gravity at the surface of the Earth? Show that in geometric units has the units of and the approximate value of

First, let us start with the comparison of the shorthand for the force of gravity as used in Classical Mechanics, , to the formal definition derived by Newton, . Evaluating this expression at the distance , we note that the difference between and is only a factor of ; which if we re-express as , we obtain the geometric expression:

We can then verify that substituting in the Earth’s values for and , that we would obtain for and that dividing by , this would give us the approximate value of .

## Part B

What is the corresponding prediction of general relativity? First, we need to decide which and we are talking about. The statement of the exercise specifies that it is the shell worker whose measurements we are to predict. Therefore, we want . Start with the result of Exercise 7:

Take the derivative of this expression with respect to , remembering that is a fixed constant. On the right side of the result, you will have a factor , where is the change in reduced circumference , not shell coordinate. Use Equation [D] to eliminate from your derivative. Then substitute for from equation [52]. Evaluate the result at to obtain the simple expression (ignoring the minus sign):

What are the limiting cases (1) as approaches and as becomes very large (but *not* infinite)?

Derivative

Equation [D]:

Limiting cases:

## Part C

The robot worker stands on a shell of radius near a black hole of mass . How many “gee”—that is, how many times the value of at Earth’s surface—is the initial acceleration of his dropped tool? What is the Newtonian prediction? (A fighter pilot risks blacking out when she makes her plane turn or rise at an acceleration of or more.)

To find the number of multiples this value is compared to , we use the relation and solve for :

So, approximately *gee-s*.

# Problem 10

# Appendix

## Deriving the velocity equation from the Bookkeeper’s View

Recall the Schwarzschild metric in polar form:

Also recall from Special Relativity, we have the expression that the stone starting at rest (its energy) is equal to its rest mass:

|  |  |
| --- | --- |
| We can take this equation and solve for and square both sides of the equation: |  |
| We then set this equal to the time-like Schwarzschild metric: |  |
| Note that in this case of a radial path, and so : |  |
| Divide through by the left-hand side: |  |
| Simplify: |  |
| Solve for : |  |

### Deriving the velocity equation with a non-zero initial velocity

|  |  |
| --- | --- |
| We can take this equation and solve for and square both sides of the equation: |  |
| We then set this equal to the time-like Schwarzschild metric: |  |
| Note that in this case of a radial path, and so : |  |
| Divide through by the left-hand side: |  |
| Simplify: |  |
| Solve for : |  |

## Deriving the velocity equation from the Shell’s view

Recall the Schwarzschild metric in polar form:

First, we must discuss each differential in terms of the shell observer, that is we must find expressions for and . This time we begin with the spacelike Schwarzschild metric and the shell observer first measures the proper distance from their shell to the next:

|  |  |
| --- | --- |
| Again, we are only concerned with a radial path and the proper distance: |  |
| Applying these substitutions and simplifying, we obtain: |  |
| On the shell, then describes : |  |

Next, we explore proper time as measured by this shell observer:

|  |  |
| --- | --- |
| Applying similar simplifications: |  |

We can then solve both expressions for and respectively. This allows us to divide the two and set them equal to the expression for velocity we derived from the Bookkeeper’s reference frame:

|  |  |
| --- | --- |
| Dividing the shell differentials: |  |
| Equating to velocity from the Bookkeeper’s reference: |  |

1. Again, using Schwarzschild geometry. We assume the black hole is rotating slowly enough to be considered stationary and spherically symmetric which allows us to choose convenient values of and . [↑](#footnote-ref-1)